The Necessary and Sufficient Conditions of Separability for Multipartite Pure States

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Dafa Li[1,*], Xiangrong Li[2], Hongtao Huang[3], Xinxin Li[4]
[1] Dept of mathematical sciences, Tsinghua University, Beijing 100084 CHINA
email:dli@math.tsinghua.edu.cn
[2] Department of Mathematics, University of California, Irvine, CA 92697-3875, USA
[3] Electrical Engineering and Computer Science Department
University of Michigan, Ann Arbor, MI 48109, USA
[4] Dept. of computer science, Wayne State University, Detroit, MI 48202, USA

Abstract

In this paper we present the necessary and sufficient conditions of separability for multipartite pure states. These conditions are very simple, and they don't require Schmidt decomposition or tracing out operations. We also give a necessary condition for a local unitary equivalence class for a bipartite system in terms of the determinant of the matrix of amplitudes and explore a variance as a measure of entanglement for multipartite pure states.

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1 Introduction:

Notation: M^+ is the complex conjugate of transpose of M.

Let $|\psi\rangle$ and $|\phi\rangle$ be two pure states of a composite system AB possessed by both Alice and Bob, where system A (B) is called Alice's (Bob's) system. By Nielsen's notation $|\psi\rangle \sim |\phi\rangle$ if and only if $|\psi\rangle$ and $|\phi\rangle$ are locally unitarily equivalent [1]. Let ρ_ψ^A and ρ_ϕ^A be the states of Alice's system. It is known that $|\psi\rangle \sim |\phi\rangle$ if and only if ρ_ψ^A and ρ_ϕ^A have the same spectrum of eigenvalues [1] [2]. A pure state is separable if and only if it can be written as a tensor product of states of different subsystems. It is also known that a state $|\psi\rangle$ of a bipartite system is separable if and only if it has Schmidt number 1 [3]. Clearly it is essential to do Schmidt decomposition to find the eigenvalues of ρ_ψ^A and ρ_ϕ^A. To obtain a Schmidt decomposition of a pure state $|\psi\rangle$, we need to compute (1) the density operator ρ_ψ^{AB}; (2) the reduced density operator ρ_ψ^A for system A; (3) the eigenvalues of ρ_ψ^A. However it is hard to compute roots of a characteristic polynomial of high degree.

Peres presented a necessary and sufficient condition for the occurrence of Schmidt decomposition for a tripartite pure state [4] and showed that the positivity of the partial transpose of a density matrix is a necessary condition for separability [5]. Thapliyal showed that a multipartite pure state is Schmidt decomposable if and only if the density matrices obtained by tracing out any

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party are separable [6]. In [7] the local invariants of quantum-bit systems were investigated. In [8][9] the local symmetry properties and local invariants of pure three-qubit states were discussed, respectively. In [10] the classification of three-qubit states was given. Bennett reported measures of multipartite pure-state entanglement in [11]. Meyer and Wallach [12] proposed a measure of $n-$qubit pure-state entanglement. Nielsen used the majorization of the eigenvalues of the reduced density operators of a composite system AB to describe the equivalence class under LOCC transformations.

For a multi $(n)-$partite system, in this paper we illustrate the reduced density operators obtained by tracing out the ith subsystem $\rho^{12...(i-1)(i+1)...n} = tr_i(\rho^{12...n}) = M_i M_i^+$, where $i = 1, 2, ..., n$ and M_i are the $d^{n-1} \times d$ matrices, of which every entry is an amplitude of the state in question. For a bipartite system AB, the reduced density operator ρ_ψ^A (ρ_ψ^B)$= MM^+$, where M is the matrix of the amplitudes. Hence $\det(\rho_\psi^A) = |\det(M)|^2$. However, for a multi $(n)-$partite system, M_i are not square. In section 2, we present a necessary and sufficient condition for separability for a bipartite system in terms of the determinants of all the 2×2 submatrices of the matrix of the amplitudes. Section 3 contains three versions of the necessary and sufficient separability criterion for a $n-$qubit system. Section 4 is devoted to study the separability of multipartite pure states, and two versions of the necessary and sufficient separability criterion are proposed. Section 5 gives a simple necessary criterion for $|\psi\rangle \sim |\phi\rangle$ for a bipartite system. Section 6 suggests an intuitive measure of multipartite pure-state entanglement.

2 The separability for a bipartite system

Let $|\psi\rangle$ be a pure state of a composite system AB possessed by both Alice and Bob. In this section we give a simple and intuitive criterion for the separability. Let $|i\rangle$ ($|j\rangle$) be the orthonormal basis for system A (B). Then we can write $|\psi\rangle = \sum_{i,j} a_{ij}|i\rangle|j\rangle$, where $\sum_{ij=0}^{n-1} |a_{ij}|^2 = 1$. Let $M = (a_{ij})_{n\times n}$ be the matrix of the amplitudes of $|\psi\rangle$. Then the criterion for the separability is as follows.

$|\psi\rangle$ is separable if and only if the determinants of all the 2×2 submatrices of M are zero.

This criterion for the separability avoids Schmidt decomposition. To compute the determinants, it needs $n^2(n-1)^2/2$ multiplication operations and $n^2(n-1)^2/4$ minus operations.

Proof. Suppose that systems A and B have the same dimension n. By definition, $|\psi\rangle$ is separable if and only if we can write $|\psi\rangle = (\sum_{i=0}^{n-1} x_i|i\rangle) \otimes (\sum_{j=0}^{n-1} y_j|j\rangle)$, where $\sum_{i=0}^{n-1} |x_i|^2 = 1$ and $= \sum_{j=0}^{n-1} |y_j|^2 = 1$. By tensor product $|\psi\rangle = \sum_{i,j=0}^{n-1} x_i y_j |i\rangle|j\rangle$. It means that $|\psi\rangle$ is separable if and only if $x_i y_j = a_{ij}$, $i, j = 0, 1, ..., (n-1)$.....(1). Let $m = \begin{pmatrix} a_{il} & a_{ik} \\ a_{jl} & a_{jk} \end{pmatrix}$ be any 2×2 submatrix of M. It is easy to check $\det(m) = a_{il}a_{jk} - a_{ik}a_{jl} = x_i y_l x_j y_k - x_i y_k x_j y_l = 0$. Therefore if $|\psi\rangle$ is separable then the determinants of all the 2×2 submatrices

of M are zero.

Conversely, suppose that the determinants of all the 2×2 submatrices of M are zero. We can write M in the block form, $M = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix} = (B_0, B_1, ..., B_{n-1})$, where A_i is the ith row and B_i is the ith column of M, respectively, $i = 0, 1, ..., (n-1)$. Let $|x_i|^2 = A_i A_i^+$.....(2) and $|y_j|^2 = B_j^+ B_j$.....(3), $i, j = 0, 1, ..., (n-1)$, respectively. Under the supposition we can show that the above x_i in (2) and y_j in (3) satisfy (1). Let us consider the case in which all the a_{ij} are real. It is not hard to extend the result to the case in which all the a_{ij} are complex. We only show $|x_0 y_0|^2 = |a_{00}|^2$ and omit the others. From (2) and (3), $|x_0 y_0|^2 = A_0 A_0^+ B_0^+ B_0 = (\sum_{j=0}^{n-1} |a_{0j}|^2)(\sum_{i=0}^{n-1} |a_{i0}|^2) = \sum_{i,j=0}^{n-1} |a_{0j}|^2 |a_{i0}|^2 = \sum_{i,j=0}^{n-1} |a_{00}|^2 |a_{ij}|^2 = |a_{00}|^2$. In the last but one step we use the equality $|a_{0j}|^2 |a_{i0}|^2 = |a_{00}|^2 |a_{ij}|^2$, which holds since $\begin{pmatrix} a_{00} & a_{0j} \\ a_{i0} & a_{ij} \end{pmatrix}$ is a 2×2 submatrix of M. This completes the proof.

Corollary

If $|\psi\rangle$ is separable then $\det(M) = 0$.

3 The separability for a $n-$qubit system

Let $|\psi\rangle$ be a pure state of a $n-$qubit system. Then we can write $|\psi\rangle = \sum_{i_1, i_2, ..., i_n \in \{0,1\}} a_{i_1 i_2 ... i_n} |i_1 i_2 ... i_n\rangle$. Let the density operator $\rho^{12...n} = |\psi\rangle\langle\psi|$ and $\rho^{12...(i-1)(i+1)...n}$ be the reduced density operator obtained by tracing out the ith qubit. Then $\rho^{12...(i-1)(i+1)...n} = tr_i(\rho^{12...n}) = M_i M_i^+$, where $i = 1, 2, ..., n$ and M_i are $2^{(n-1)} \times 2$ matrices of the form $\left(a_{b_1 b_2 ... b_{i-1} 0 b_{i+1} ... b_n}, a_{b_1 b_2 ... b_{i-1} 1 b_{i+1} ... b_n}\right)$ in which b_1,b_2,...,$b_n \in \{0, 1\}$.

For example, let $|\psi\rangle$ be a state of a 3-qubit system. Then $|\psi\rangle$ can be written as $|\psi\rangle = \sum_{i=0}^{7} a_i |i\rangle$. M_3 is a 4×2 matrix $\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \\ a_4 & a_5 \\ a_6 & a_7 \end{pmatrix}$. Each entry of M_3 is an amplitude of $|\psi\rangle$.

There are three versions of the separability.

Version 1. $|\psi\rangle$ is separable if and only if the determinants of all the 2×2 submatrices of M_1,M_2,.... and M_n are zero.

The proof of version 1 is similar to the one for a bipartite system in section 2.

Version 2. $|\psi\rangle$ is separable if and only if $a_i a_j = a_k a_l$, where $i + j = k + l$ and $i \oplus j = k \oplus l$ where $0 \leq i, j, k, l \leq 2^n - 1$ are $n-$bit strings and $\oplus$ indicates addition modulo 2.

For example, 2, 7, 5 and 4 can be written in binary numbers as $010, 111, 101$ and 100, respectively. It is well known 010 + 111(modulo 2)= 101, 101 +

$100 = 001$(modulo 2). Therefore $2+7 \neq 5+4$(modulo 2) though $2+7 = 5+4 = 9$.

Using this condition it is easy to verify that states $|W\rangle = 1/\sqrt{n}(|2^0\rangle + |2^1\rangle + ... + |2^{n-1}\rangle$ and $|GHZ\rangle = 1/\sqrt{2}(|0^{(n)}\rangle + |1^{(n)}\rangle)$ for a $n-$qubit system [13] are entangled.

Let $i_1i_2...i_n$, $j_1j_2...j_n$, $k_1k_2...k_n$ and $l_1l_2...l_n$ be $n-$bit strings of i,j,k and l, respectively. Then version 3 is phrased below.

Version 3. $|\psi\rangle$ is separable if and only if $a_ia_j = a_ka_l$, where $\{i_t, j_t\} = \{k_t, l_t\}$, $t = 1, 2, ..., n$.

The following lemma 1 shows that versions 2 and 3 are equivalent to each other.

Lemma 1. $i + j = k + l$ and $i \oplus j = k \oplus l$ if and only if $\{i_t, j_t\} = \{k_t, l_t\}$, $t = 1, 2, ..., n$.

The proof of lemma 1 is put in appendix A.

We argue version 3 next.

Assume that $|\psi\rangle = (x_0^{(1)}|0\rangle + x_1^{(1)}|1\rangle) \otimes (x_0^{(2)}|0\rangle + x_1^{(2)}|1\rangle) \otimes ... \otimes (x_0^{(n)}|0\rangle + x_1^{(n)}|1\rangle)$. By tensor product $x_{i_1}^{(1)}x_{i_2}^{(2)}....x_{i_n}^{(n)} = a_{i_1i_2...i_n}$, where $i_t = 0, 1$, $t = 1, 2, ..., n$. Then $a_ia_j = x_{i_1}^{(1)}x_{j_1}^{(1)}x_{i_2}^{(2)}x_{j_2}^{(2)}....x_{i_n}^{(n)}x_{j_n}^{(n)}$ and $a_ka_l = x_{k_1}^{(1)}x_{l_1}^{(1)}x_{k_2}^{(2)}x_{l_2}^{(2)}....x_{k_n}^{(n)}x_{l_n}^{(n)}$. Explicitly, $a_ia_j = a_ka_l$ whenever $\{i_t,j_t\} = \{k_t, l_t\}$, $t = 1, 2, ..., n$.

Conversely, suppose that $a_ia_j = a_ka_l$ whenever $\{i_t, j_t\} = \{k_t, l_t\}$, $t = 1, 2, ..., n$. Let $\left|x_{i_t}^{(t)}\right|^2 = \sum_{i_1,..,i_{t-1},i_{t+1},..,i_n \in \{0,1\}} |a_{i_1i_2,...,i_n}|^2$, where $t = 1, 2, ..., n$. We can show $|x_{i_1}^{(1)}x_{i_2}^{(2)}....x_{i_n}^{(n)}|^2 = |a_{i_1i_2...i_n}|^2$. We only demonstrate the cases of $n = 2$ and 3 to give the essential ideas of the general case.

When $n = 2$, see section 2. When $n = 3$, see appendix B. The two cases suggest that it be simpler to prove $|x_{i_1}^{(1)}x_{i_2}^{(2)}....x_{i_n}^{(n)}|^2 = |a_{i_1i_2...i_n}|^2 \left(\sum |a_{i_1i_2...i_n}|^2\right)^{n-1}$. Now we finish the argument for the real number case. It is not hard to extend the result to the complex number case.

4 The separability for a multi $(n)-$partite system

Assume that each subsystem has the same dimension d. Let $|i_t\rangle$ be the orthonormal basis $|0\rangle$,$|1\rangle$,...,$|(d-1)\rangle$ for the tth subsystem. Then any pure state $|\psi\rangle$ can be written as $|\psi\rangle = \sum_{i_1,i_2,...,i_n=0}^{d-1} a_{i_1i_2...i_n}|i_1i_2...i_n\rangle$. Assume that $|\psi\rangle$ is separable. Then we can write $|\psi\rangle = \left(\sum_{i_1=0}^{d-1} x_{i_1}^{(1)}|i_1\rangle\right) \otimes \left(\sum_{i_2=0}^{d-1} x_{i_2}^{(2)}|i_2\rangle\right) \otimes ... \otimes \left(\sum_{i_n=0}^{d-1} x_{i_n}^{(n)}|i_n\rangle\right)$. By tensor product $x_{i_1}^{(1)}x_{i_2}^{(2)}....x_{i_n}^{(n)} = a_{i_1i_2...i_n}$, where i_1,i_2,...,$i_n \in \{0, 1, ..., (d-1)\}$.

Let the density operator $\rho^{12...n} = |\psi\rangle\langle\psi|$ and $\rho^{12...(i-1)(i+1)...n}$ be the reduced density operator obtained by tracing out the ith subsystem. Then $\rho^{12...(i-1)(i+1)...n} = tr_i(\rho^{12...n}) = M_iM_i^+$, where $i = 1, 2, ..., n$ and M_i are $d^{n-1} \times d$ matrices of the amplitudes of the form
$\left(a_{k_1k_2...k_{i-1}0k_{i+1}...k_n}, a_{k_1k_2...k_{i-1}1k_{i+1}...k_n}, ..., a_{k_1k_2...k_{i-1}(d-1)k_{i+1}...k_n}\right)$, where k_1,k_2,...,k_{i-1},k_{i+1},...,$k_n \in \{0, 1, ..., (d-1)\}$.

There are two versions of the separability.

Version 1. $|\psi\rangle$ is separable if and only if the determinants of all the 2×2 submatrices of M_1, M_2, ... and M_n are zero.

Version 2. $|\psi\rangle$ is separable if and only if $a_{i_1 i_2 ... i_n} a_{j_1 j_2 ... j_n} = a_{k_1 k_2 ... k_n} a_{l_1 l_2 ... l_n}$, where $\{i_t, j_t\} = \{k_t, l_t\}$, $t = 1, 2, ..., n$.

The proof of version 1 is similar to the one for a bipartite system. The proof of version 2 is similar to the one for a $n-$qubit system.

When $n = 2$, the criterion is reduced to the one for a bipartite system. When $d = 2$, the criterion is reduced to the one for a n-qubit system.

5 A necessary condition for a local unitary equivalence class for a bipartite system

We use the following lemma 2 to establish the necessary condition.

Lemma 2. Let $|\psi\rangle$ be a pure state of a composite system AB possessed by both Alice and Bob. Let $M = (a_{jk})_{n \times n}$ be the matrix of the amplitudes of $|\psi\rangle$. Let $\rho^{AB} = |\psi\rangle\langle\psi|$ and $\rho^A = tr_B(\rho^{AB})$. Then $|\det(M)|^2$ is just the product of the eigenvalues of ρ^A.

The proof is put in appendix C.

Lemma 2 reveals the relation between the determinant of the matrix of the amplitudes and the eigenvalues of ρ^A for a bipartite system.

The corollary of lemma 2

Let M_ψ (M_ϕ) be the matrix of the amplitudes of a pure state $|\psi\rangle$ ($|\phi\rangle$) of a composite system AB. Then $|\det(M_\psi)| = |\det(M_\phi)|$ whenever $|\psi\rangle \sim |\phi\rangle$. That is, $|\det(M_\psi)|$ is invariant under local unitary operators.

It is well known that it only needs $O(n^3)$ multiplication operations to compute $|\det(M)|$ instead of doing Schmidt decomposition in [1][2].

For a two-qubit system, let $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ and $\rho^{12} = |\psi\rangle\langle\psi|$. By lemma 2 $|ad - bc|^2$ is the product of the eigenvalues of ρ^1. Let $|ad - bc| = \in$. We can show that $\in$ satisfies $0 \leq \in \leq \frac{1}{2}$ and eigenvalues $\lambda_\pm = \frac{1 \pm \sqrt{1 - 4\in^2}}{2}$. Hence, $|\psi\rangle \sim \sqrt{\lambda_+}|00\rangle + \sqrt{\lambda_-}|11\rangle$ or $|\psi\rangle \sim \sqrt{\lambda_-}|00\rangle + \sqrt{\lambda_+}|11\rangle$.

6 The variance as a measure of entanglement

We obtain the necessary and sufficient conditions of separability in sections 2, 3 and 4. Apparently, $|a_{i_1 i_2 ... i_n} a_{j_1 j_2 ... j_n} - a_{k_1 k_2 ... k_n} a_{l_1 l_2 ... l_n}|$, where $\{i_t, j_t\} = \{k_t, l_t\}$, $t = 1, 2, ..., n.$, is just a deviation from a product state. It is intuitive to suggest the variance: $\sum |a_{i_1 i_2 ... i_n} a_{j_1 j_2 ... j_n} - a_{k_1 k_2 ... k_n} a_{l_1 l_2 ... l_n}|^2$, where $\{i_t, j_t\} = \{k_t, l_t\}$, $t = 1, 2, ..., n$, as a measure of entanglement of $|\psi\rangle$. Let $D_E(|\psi\rangle)$ be the measure of entanglement.

$D_E(|\psi\rangle)$ has the following properties.

Property 1. $D_E(|\psi\rangle) = 0$ if and only if $|\psi\rangle$ is separable.

The properties for a two-qubit system

For a two-qubit system, let $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$. Then $D_E(|\psi\rangle) = |ad - bc|^2$.

Property 2. The maximum of $D_E(|\psi\rangle) = |ad - bc|^2 \leq (|ad| + |bc|)^2 \leq (\frac{|a|^2+|d|^2}{2} + \frac{|b|^2+|c|^2}{2})^2 = \frac{1}{4}$.

When a, b, c and d are real, by computing extremum it is derived that the maximally entangled states must be of the forms: $x|00\rangle + y|01\rangle - y|10\rangle + x|11\rangle$ or $x|00\rangle + y|01\rangle + y|10\rangle - x|11\rangle$.

Property 3. $|\psi\rangle \sim |\psi'\rangle$ if and only if $D_E(|\psi\rangle) = D_E(|\psi'\rangle)$.

Given $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ and $|\psi'\rangle = a'|00\rangle + b'|01\rangle + c'|10\rangle + d'|11\rangle$. Suppose that $|\psi\rangle \sim |\psi'\rangle$. By the necessary condition in section 5, $D_E(|\psi\rangle) = D_E(|\psi'\rangle)$.

Conversely, suppose $D_E(|\psi\rangle) = D_E(|\psi'\rangle)$. Let us show $|\psi\rangle \sim |\psi'\rangle$. Using Schmidt decomposition, we can write $|\psi\rangle \sim \sqrt{\lambda_1}|00\rangle + \sqrt{\lambda_2}|11\rangle$, where $\lambda_1 + \lambda_2 = 1$. As discussed above $|ad - bc| = \sqrt{\lambda_1}\sqrt{\lambda_2}$. As well using Schmidt decomposition we can write $|\psi'\rangle \sim \sqrt{\rho_1}|00\rangle + \sqrt{\rho_2}|11\rangle$, where $\rho_1 + \rho_2 = 1$, and $|a'd' - b'c'| = \sqrt{\rho_1}\sqrt{\rho_2}$. Thus $\lambda_1\lambda_2 = \rho_1\rho_2$. Then $\lambda_1(1-\lambda_1) = \rho_1(1-\rho_1)$. There are two cases. Case 1. $\lambda_1 = \rho_1$. Then $\lambda_2 = \rho_2$. Case 2. $\lambda_1 + \rho_1 + 1 = 0$. In the case $\lambda_2 = \rho_1$ and $\lambda_1 = \rho_2$. It means that $|\psi\rangle$ and $|\psi'\rangle$ have the same Schmidt co-efficient for either of the two cases. By factor 5 in [1][2], $|\psi\rangle \sim |\psi'\rangle$.

Nielsen in [1] showed $|\psi'\rangle \sim |\psi''\rangle$ by calculating eigenvalue, where $|\psi'\rangle = \sqrt{\alpha_+}|00\rangle + \sqrt{\alpha_-}|11\rangle$, and $|\psi''\rangle = (|00\rangle + |1\rangle(\cos\gamma|0\rangle + \sin\gamma|1\rangle))/\sqrt{2}$. By property 3 it only needs to check $\sqrt{\alpha_+}\sqrt{\alpha_-} = \sin\gamma/2$.

Conclusion

In this paper we have presented the necessary and sufficient conditions of separability for multipartite pure states. These conditions don't require Schmidt decomposition or tracing out operations. By using the conditions it is easy to check whether or not a multipartite pure state is entangled.

Appendix A. The proof of lemma 1

Let $\alpha_1\alpha_2...\alpha_n$, $\beta_1\beta_2...\beta_n$, $\delta_1\delta_2...\delta_n$ and $\gamma_1\gamma_2...\gamma_n$ be the $n-$bit strings of α, β, δ and γ, respectively.

Lemma 1. $\{\alpha_i, \beta_i\}=\{\delta_i, \gamma_i\}$, $i = 1, 2, ..., n$, if and only if $\alpha + \beta = \delta + \gamma$ and $\alpha \oplus \beta = \delta \oplus \gamma$, where $\oplus$ indicates addition modulo 2.

Proof. Suppose $\{\alpha_i, \beta_i\}=\{\delta_i, \gamma_i\}$, $i = 1, 2, ..., n$. Since $\alpha + \beta = (\alpha_1 + \beta_1)2^{n-1} + (\alpha_2 + \beta_2)2^{n-2} + ... + (\alpha_n + \beta_n)$ and $\delta + \gamma = (\delta_1 + \gamma_1)2^{n-1} + (\delta_2 + \gamma_2)2^{n-2} + ... + (\delta_n + \gamma_n)$, by the supposition it is easy to see $\alpha + \beta = \delta + \gamma$. It is straightforward to obtain $\alpha_1\alpha_2...\alpha_n \oplus \beta_1\beta_2...\beta_n = \delta_1\delta_2...\delta_n \oplus \gamma_1\gamma_2...\gamma_n$.

Conversely, suppose $\alpha + \beta = \delta + \gamma$ and $\alpha \oplus \beta = \delta \oplus \gamma$. First let us consider the case where $n = 1$. There are three cases.

Case 1. $\alpha_1 + \beta_1 = \delta_1 + \gamma_1 = 0$. This means $\alpha_1 = \beta_1 = \delta_1 = \gamma_1 = 0$.

Case 2. $\alpha_1 + \beta_1 = \delta_1 + \gamma_1 = 1$. This implies $\{\alpha_1, \beta_1\} = \{\delta_1, \gamma_1\} = \{1, 0\}$.

Case 3. $\alpha_1 + \beta_1 = \delta_1 + \gamma_1 = 2$. This says $\alpha_1 = \beta_1 = \delta_1 = \gamma_1 = 1$.

No matter which of the above three cases happens, it yields $\{\alpha_1, \beta_1\}=\{\delta_1, \gamma_1\}$.

Let us consider the case n. Since $\alpha + \beta = \delta + \gamma$, $(\alpha_1 + \beta_1)2^{n-1} + (\alpha_2 + \beta_2)2^{n-2} + ... + (\alpha_n + \beta_n) = (\delta_1 + \gamma_1)2^{n-1} + (\delta_2 + \gamma_2)2^{n-2} + ... + (\delta_n + \gamma_n)$. Again

since $\alpha \oplus \beta = \delta \oplus \gamma$, that is, $\alpha_1\alpha_2...\alpha_n \oplus \beta_1\beta_2...\beta_n = \delta_1\delta_2...\delta_n \oplus \gamma_1\gamma_2...\gamma_n$, we obtain $\alpha_i \oplus \beta_i = \delta_i \oplus \gamma_i$, $i = 1, 2, ..., n$. There are two cases.

Case 1. $\alpha_n \oplus \beta_n = \delta_n \oplus \gamma_n = 1$. In the case $\{\alpha_n, \beta_n\} = \{\delta_n, \gamma_n\} = \{0, 1\}$. Then $(\alpha_1 + \beta_1)2^{n-2} + (\alpha_2 + \beta_2)2^{n-3} + ... + (\alpha_{n-1} + \beta_{n-1}) = (\delta_1 + \gamma_1)2^{n-2} + (\delta_2 + \gamma_2)2^{n-3} + ... + (\delta_{n-1} + \gamma_{n-1})$ and $\alpha_i \oplus \beta_i = \delta_i \oplus \gamma_i$, $i = 1, 2, ..., n-1$. By induction hypothesis $\{\alpha_i, \beta_i\} = \{\delta_i, \gamma_i\}$, $i = 1, 2, ..., n-1$.

Case 2. $\alpha_n \oplus \beta_n = \delta_n \oplus \gamma_n = 0$. There are two subcases.

Subcase 2.1. $\alpha_n = \beta_n = \delta_n = \gamma_n = 0$ or $\alpha_n = \beta_n = \delta_n = \gamma_n = 1$. As discussed in case 1, we can obtain $\{\alpha_i, \beta_i\} = \{\delta_i, \gamma_i\}$, $i = 1, 2, ..., n-1$ by induction hypothesis.

Subcase 2.2. $\alpha_n = \beta_n = 1$ and $\delta_n = \gamma_n = 0$ or $\alpha_n = \beta_n = 0$ and $\delta_n = \gamma_n = 1$. Let us consider the former case. In the case $(\alpha_1 + \beta_1)2^{n-2} + (\alpha_2 + \beta_2)2^{n-3} + ... + (\alpha_{n-2} + \beta_{n-2})2 + (\alpha_{n-1} + \beta_{n-1} + 1) =$

$(\delta_1 + \gamma_1)2^{n-2} + (\delta_2 + \gamma_2)2^{n-3} + ... + (\delta_{n-2} + \gamma_{n-2})2 + (\delta_{n-1} + \gamma_{n-1})$.

Since $\alpha_{n-1} \oplus \beta_{n-1} = \delta_{n-1} \oplus \gamma_{n-1}$, either $\alpha_{n-1} \oplus \beta_{n-1} = \delta_{n-1} \oplus \gamma_{n-1} = 0$ or 1 causes that one of $(\alpha_{n-1} + \beta_{n-1} + 1)$ and $(\delta_{n-1} + \gamma_{n-1})$ is odd and the other is even. It contradicts $\alpha \oplus \beta = \delta \oplus \gamma$.

Appendix B. The separability for a $n-$qubit system

When $n = 3$, let us show $|x^{(1)}_{i_1} x^{(2)}_{i_2} x^{(3)}_{i_3}|^2 = |a_{i_1 i_2 i_3}|^2$ when $a_i a_j = a_k a_l$, where $\{i_t, j_t\} = \{k_t, l_t\}$, $t = 1, 2, 3$. We only illustrate $|x^{(1)}_0 x^{(2)}_0 x^{(3)}_0|^2 = |a_{000}|^2$. Other cases then follow readily. Experientially, it is simpler to prove $|x^{(1)}_0 x^{(2)}_0 x^{(3)}_0|^2 = |a_{000}|^2(\sum_{i,j,k\in\{0,1\}} |a_{ijk}|^2)(\sum_{i,j,k\in\{0,1\}} |a_{ijk}|^2)$, where $|x^{(1)}_0|^2 = \sum_{i,j\in\{0,1\}} |a_{0ij}|^2$, $|x^{(2)}_0|^2 = \sum_{k,l\in\{0,1\}} |a_{k0l}|^2$ and $|x^{(3)}_0|^2 = \sum_{p,q\in\{0,1\}} |a_{pq0}|^2$.

First we show that $a_{0ij}a_{k0l}a_{pq0}$ can be rewritten as $a_{000}a_{\alpha_1\alpha_2\alpha_3}a_{\delta_1\delta_2\delta_3}$. There are the following four cases.

Case 1. Consider $a_{0ij}a_{k0l}$ and the pairs $\{0, k\}, \{i, 0\}$ and $\{j, l\}$. If $j * l = 0$, then $a_{0ij}a_{k0l} = a_{000}a_{ki(j+l)}$ since $\{j, l\} = \{0, j + l\}$.

Case 2. Consider $a_{0ij}a_{pq0}$ and the pairs $\{0, p\}, \{i, q\}$ and $\{j, 0\}$. If $i * q = 0$, then $a_{0ij}a_{pq0} = a_{000}a_{p(i+q)j}$ since $\{i, q\} = \{0, i + q\}$.

Case 3. Consider $a_{k0l}a_{pq0}$ and the pairs $\{k, p\}, \{0, q\}$ and $\{l, 0\}$. If $k * p = 0$, then $a_{k0l}a_{pq0} = a_{000}a_{(k+p)ql}$ since $\{k, p\} = \{0, k + p\}$.

Case 4. Otherwise $i = j = l = k = p = q = 1$. It is not hard to derive $a_3a_5a_6 = a_1a_7a_6 = a_0a_7^2$.

Second, let us show that $a_{000}a_{\alpha_1\alpha_2\alpha_3}a_{\delta_1\delta_2\delta_3}$ can be rewritten as $a_{0ij}a_{k0l}a_{pq0}$. If $a_{000}a_{\alpha_1\alpha_2\alpha_3}a_{\delta_1\delta_2\delta_3}$ is of the forms: $a_{000}a_{0ij}a_{k0l}$, $a_{000}a_{0ij}a_{pq0}$ or $a_{000}a_{k0l}a_{pq0}$, then these forms are desired. Otherwise $a_{000}a_{\alpha_1\alpha_2\alpha_3}a_{\delta_1\delta_2\delta_3}$ must be $a_0a_6a_6$, $a_0a_3a_3$, $a_0a_5a_5$ or of the form $a_0a_7a_{rst}$, which can be rewritten as $a_2a_4a_6$, $a_1a_2a_3$, $a_1a_4a_5$, $a_1a_6a_{rst}$, respectively. $a_2a_4a_6$, $a_1a_2a_3$ and $a_1a_4a_5$ are just desired and $a_1a_6a_{rst}$ is furthermore rewritten as follows. There are three cases.

Case 1. In the case $r = 0$ or $s = 0$, this is desired.

Case 2. In the case $r = s = t = 1$, $a_1a_6a_7 = a_3a_5a_6$, desired.

Case 3. In the case $r = s = 1$ and $t = 0$, $a_1a_6a_6 = a_2a_5a_6$, desired.

Appendix C. The proof of lemma 2

Proof. Suppose that systems A and B have the same dimensions n. Let $|\psi\rangle =$

$\sum_{i,j=0}^{n-1} a_{ij}|i\rangle|j\rangle$. Then $M = (a_{ij})_{n\times n}$. Let density operator $\rho^{AB} = |\psi\rangle\langle\psi|$. Then $\rho^{AB} = (\sum_{i,j=0}^{n-1} a_{ij}|i\rangle|j\rangle)(\sum_{l,k=0}^{n-1} a^*_{lk}\langle l|\langle k|) = \sum_{i,j=0}^{n-1}\sum_{l,k=0}^{n-1} a_{ij}a^*_{lk}|i\rangle|j\rangle\langle l|\langle k|$

$= \sum_{i,l=0}^{n-1}\sum_{j,k=0}^{n-1} a_{ij}a^*_{lk}|i\rangle|j\rangle\langle l|\langle k|$. The reduced density operator for system A is defined by $\rho^A = tr_B(\rho^{AB})$. Let us compute ρ^A.

$\rho^A = \sum_{i,l=0}^{n-1}\sum_{j,k=0}^{n-1} a_{ij}a^*_{lk}|i\rangle\langle l|\delta_{kj}$ (where $\delta_{kj} = 1$ when $k = j$. Otherwise 0.) $= \sum_{i,l=0}^{n-1}\sum_{j=0}^{n-1} a_{ij}a^*_{lj}|i\rangle\langle l| = \sum_{i,l=0}^{n-1}(\sum_{j=0}^{n-1} a_{ij}a^*_{lj})|i\rangle\langle l|$. Let $A_i = (a_{i0}, a_{i1}, ...a_{i(n-1)})$, that is, the ith row of A. Then $\sum_{j=0}^{n-1} a_{ij}a^*_{lj} = A_iA^+_l$. Finally $\rho^A = \sum_{i,l=0}^{n-1} A_iA^+_l|i\rangle\langle l| = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n-1} \end{pmatrix}(A^+_0, A^+_1, ..., A^+_{n-1}) = MM^+$. Thus $\det(\rho^A) = |\det(M)|^2$. Hence $|\det(M)|^2$ is just the product of the eigenvalues of ρ^A. Q.E.D.

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